Linear Horistologies

Trandafir BĂLAN and Dana CONSTANTINESCU*

Abstract

The aim of the work is to investigate the relations between linearity and horistology in order to generalize the linear spaces endowed with super-additive norms, like the Minkowskian space-times.

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Introduction. According to our previous paper [1], a **horistology** on the non-void set W (also called *world* if inspired by Relativity) is a function

$$\chi : W \to \mathcal{P}(\mathcal{P}(W))$$

for which:
$[h_1]$ $x \notin P$ for all $x \in W$ and $P \in \chi(x)$;
$[h_2]$ $P \in \chi(x), Q \subseteq P \Longrightarrow Q \in \chi(x)$;
$[h_3]$ $P, Q \in \chi(x) \Longrightarrow P \cup Q \in \chi(x)$;
$[h_4]$ $\forall P \in \chi(x), \exists Q \in \chi(x)$ such that $[y \in P$ and $R \in \chi(y)] \Longrightarrow [R \subseteq Q]$.

The pair (W, χ) represents a *horistological world* (or *space*, briefly *h.w.*). The elements of $\chi(x)$ are called *perspectives* of x. Obviously, $\chi(x)$.is an ideal of subsets of W. It is easy to prove that if (W, χ) is a h.w., then

$$K(\chi) = \{(x, y) : \{y\} \in \chi(x)\} \cup \delta$$

is an order on W (called *causality* in space-time).

The most important examples of h.w., including the Minkowskian space-time, are linear spaces endowed with super-additive norms. Such a norm (also called *timer* when it measures time) is a functional

$$p : K[0] \to \mathbb{R}_+$$

*University of Craiova, Romania

for which:

[t_1] $p(x) = 0$ if and only if $x = 0$

[t_2] $p(\lambda x) = \lambda p(x)$ for all $\lambda \in \mathbb{R}_+$ and $x \in K[0]$

[t_3] $p(x + y) \geq p(x) + p(y)$ for all $x, y \in K[0]$,

where K is a linear order on the real linear space W.

In this case, for each $x \in W$, $\chi(x)$ is defined by the ideal base

$$\beta(x) = \{P \subset W : \exists r > 0 \ \ \textit{such that } P \subseteq H(x, r)\}$$

where

$$H(x, r) = \{y \in W : p(y - x) > r\}$$

It is also remarkable that $K = K(\chi)$.

By analogy with the classical theory of the linear topological spaces (see [2]), where the operations are continuous functions, for the horistological worlds we will naturally ask these operations to be discrete (in the sense of [1]). In order to explain this condition, we recall that a function $f : W \to V$, where (W, χ) and (V, ψ) are h.w., is said to be *discrete* at x if

$$f(\chi(x)) \subseteq \psi(f(x)).$$

In addition, because the operations of a linear space are defined on Cartesian products, we mention that if (W_1, χ_1) and (W_2, χ_2) are horistological worlds, and $W = W_1 \times W_2$, then the *product* horistology on W is defined by

$$\gamma(x_1, x_2) = \{P_1 \times P_2 : P_i \in \chi_i(x_i), i = 1, 2\},$$

which represents its ideal base.

Because the h.w. are essentially ordered sets, we deal only with real linear spaces. In particular, the *standard* horistology of $\mathbb{R}$ is generated by the super-additive norm $p(x) = x$, defined on $\mathbb{R}_+$, i.e. on the cone of the usual order of $\mathbb{R}$. This horistology will be noted σ.

Example 1 *Let W be a real linear space, K be a linear order on W, and let $p : K[0] \to \mathbb{R}_+$ be a super-additive norm. If χ denotes the horistology generated by p, then the operations of W are discrete functions relative to χ. More precisely, for every $x_1, x_2 \in W$, $P_1 \in \chi(x_1)$ and $P_2 \in \chi(x_2)$, we have $P_1 + P_2 \in \chi(x_1 + x_2)$. In fact, by hypothesis there exist $r_1, r_2 > 0$ such that $P_i \subseteq H(x_i, r_i)$, $i = 1, 2$, hence for every $y_i \in P_i$, $i = 1, 2$, in accordance with [t_3], we have*

$$p(y_1 + y_2 - (x_1 + x_2)) > r_1 + r_2.$$

Similarly, the product with real numbers is a discrete function on $\mathbb{R}_+ \times K[0]$*, as a consequence of* $[t_2]$*. In fact, if we consider* $P \in \sigma(\lambda)$*, and* $Q \in \chi(x)$*, this means that* $P \subseteq H_{\mathbb{R}}(\lambda, \varepsilon)$ *and* $Q \subseteq H_W(x, r)$ *for some* $\varepsilon > 0$ *and* $r > 0$*. If* $\lambda \geq 0$*,* $(0, x) \in K$*,* $\mu > \varepsilon + \lambda$*, and* $p(y - x) > r$*, then it follows that*

$$p(\mu y - \lambda x) \geq \mu p(y - x) + (\mu - \lambda)p(x) > \varepsilon r,$$

hence $PQ \subseteq H_W(\lambda x, \varepsilon r)$*, and finally* $PQ \in \chi(\lambda x)$*.*

This example contains many particular cases, namely the Minkowskian space-time, the event spaces, as well as $\mathbb{R}$, $C_{\mathbb{R}}([a, b])$, etc. In all these spaces, besides addition and multiplication, we may similarly see that translation, dilation and embedding are discrete functions too.

Definition 2 *Let* W *be a real linear space, and let* χ *be a horistology on* W*. We say that* χ *is a* ***linear*** *horistology (**compatible** with linearity, etc.) if:*

(*i*) *For every* $x \in W$*, the translation* $T_x : W \to W$*, defined by*

$$T_x(y) = x + y,$$

is discrete on W*.*

(*ii*) *The addition* $A : W \times W \to W$*, defined by*

$$A(x, y) = x + y,$$

is discrete at $(0, 0)$*.*

(*iii*) *For every* $\lambda > 0$*, the dilation* $D_\lambda : W \to W$*, of values*

$$D_\lambda(x) = \lambda x,$$

is discrete at 0*.*

(*iv*) *For all* $x \in K[0]$*, the embedding of the real line* $E_x : \mathbb{R} \to W$*,*

$$E_x(\lambda) = \lambda x,$$

is discrete at 0*.*

(*v*) *The multiplication* $M : \mathbb{R} \times W \to W$*, defined by*

$$M(\lambda, x) = \lambda x,$$

is discrete at $(0, 0)$*.*

In this case, the pair (W, χ) will be called *linear horistological world* (briefly *l.h.w.*).

Proposition 3 *In every l.h.w. we have:*

a) The addition is discrete on $W \times W$*.*

b) The multiplication is discrete on $\mathbb{R}_+ \times K[0]$*.*

Proof. *a) From (i) we deduce that for all* $P \in \chi(0)$ *and* $x \in W$ *we have* $x + P \in \chi(x)$*. Because* $T_x^{-1} = T_{-x}$ *is also discrete on* W*, and particularly at* x*, we obtain*

$$\chi(x) = x + \chi(0). \tag{1}$$

On the other hand, the discreteness of A *at* $(0,0)$ *means that* $P + Q \in \chi(0)$ *whenever* $P, Q \in \chi(0)$*. Now, in order to prove the assertion a), it is enough to remark that for all* $x, y \in W$*,* $P \in \chi(x)$*, and* $Q \in \chi(y)$*, the identity*

$$P + Q = T_{x+y}[A(T_{-x}(P), T_{-y}(Q)]$$

holds, hence $P + Q$ *is a perspective of* $x + y$*.*

b) Based on the relation

$$\mu y - \lambda x = (\mu - \lambda)(y - x) + \lambda(y - x) + (\mu - \lambda)x,$$

we deduce that for all $\lambda \in \mathbb{R}_+$*,* $x \in K[0]$*,* $P \in \chi(0)$*, and* $Q \in \sigma(\lambda)$*, we have*

$$QP \subseteq T_{\lambda x}\{A\left\langle A[D_\lambda(T_{-x}(P)), E_x(Q - \lambda)], M(Q - \lambda, T_{-x}(P))\right\rangle\}.$$

According to the hypothesis $(i) - (v)$*, all the involved functions are discrete, hence* $QP \in \chi(\lambda x)$*.* ■

Corollary 4 *If* (W, χ) *is a l.h.w., then:*

a) The horistology χ *is uniform.*

b) The order $K(\chi)$ *is linear.*

c) If (V, ψ) *is another l.h.w., then a linear operator* $U : W \to V$ *is discrete on* W *if and only if it is discrete at* 0*.*

Proof. *a)* To each perspective $P \in \chi(0)$ we attach the prospect

$$\pi_P = \{(x, y) : y \in x + P\}.$$

It is easy to see that the family of all such prospects represents an ideal base for a uniform horistology on W, in the sense of [1].

b) Because of (1), for any $x \in W$ we also have

$$K[x] = x + K[0],$$

i.e. K is invariant under translations. Using (*iii*). we similarly see that $(x, y) \in K$ implies $(\lambda x, \lambda y) \in K$, whenever $\lambda \geq 0$.

c) Since U is linear, for every $x \in W$ we have

$$U = T_x \circ U \circ T_{-x},$$

which reduces the discreteness of U at x to its discreteness at 0. ■

In the rest of the paper we will analyze some specific properties of the perspectives $P \in \chi(0)$ in a l.h.w.

Definition 5 *Let W be a real linear space. We say that the set $A \subseteq W$ is* ***extensive*** *if $\alpha A \subseteq A$ whenever $\alpha \geq 1$. For an arbitrary subset X of W we define the* ***extension*** *$e(X)$ by the formula*

$$e(X) = \cup\{\alpha X : \alpha \geq 1\}.$$

Relative to the *extension operator* $e : \mathcal{P}(W) \to \mathcal{P}(W)$ we mention the following properties:

Proposition 6 *If W is a real linear space, then:*

1) *For every $X \subseteq W$, $e(X)$ is extensive.*
2) *A is extensive if and only if $A = e(A)$.*
3) *$A \subseteq B$ implies $e(A) \subseteq e(B)$*
4) *$e(A \cup B) = e(A) \cup e(B)$ for all $A, B \subseteq W$.*
5) *$e(A \cap B) \subseteq e(A) \cap e(B)$ for all $A, B \subseteq W$.*

The proof is routine and will be omitted.

Lemma 7 *If (W, χ) is a l.h.w., then $\chi(0)$ admits an ideal base consisting of extensive prospects.*

Proof. Because $X \subseteq e(X)$ for all $X \subseteq W$, it is sufficient to show that $e(P) \in \chi(0)$ holds for all $P \in \chi(0)$. In fact, using the discreteness of the multiplication at $(0, 0)$, if we note

$$Q = \{\alpha \in \mathbb{R} : \alpha \geq 1\},$$

we obtain $e(P) = M(Q, P) \in \chi(0)$. ■

Remark 8 *The proof of the above Lemma may also be based on the relation between $e(P)$ and $K[P]$, where $K = K(\chi)$, and $P \in \chi(0)$. More precisely, we have:*

a) For all $P \in \chi(0)$, the set $K[P]$ is extensive.

b) For every $P \in \chi(0)$, the inclusion $e(P) \subseteq K[P]$ holds.

In fact, Lemma 7 follows by combining these properties with the general fact that

$$\{K[P] : P \in \chi(x)\}$$

is an ideal base of $\chi(x)$, where $x \in W$ is arbitrary (see also [1]).

In order to prove a), we recall that

$$K[P] = \{y \in W : (x, y) \in K \quad for \quad some \quad x \in P\}.$$

If $z \in e(K[P])$, then there exist $y \in K[P]$ and $\alpha \geq 1$ such that $z = \alpha y$. But in this case $(y, \alpha y) \in K$ too, hence using the membership $(x, y) \in K$ for some $x \in P$, we deduce that $(x, z) \in K$. Consequently, $z \in K[P]$. This proves that $e(K[P]) \subseteq K[P]$, i.e. $K[P]$ is extensive.

The second property follows from the former one, and from the monotony of e, since $P \subseteq K[P]$ implies

$$e(P) \subseteq e(K[P]) = K[P].$$

Theorem 9 *If (W, χ) is a l.h.w., then there exists a family $\mathcal{B} \subseteq \chi(0)$ such that:*

[lh_1] $0 \notin \overline{P}$ whenever $\overline{P} \in \mathcal{B}$.

[lh_2] For every $\overline{P}, \overline{Q} \in \mathcal{B}$ there is $\overline{L} \in \mathcal{B}$ such that $\overline{P} \cup \overline{Q} \subseteq \overline{L}$.

[lh_3] Every $\overline{P} \in \mathcal{B}$ is extensive.

[lh_4] For every $\overline{P} \in \mathcal{B}$ there is $\overline{S} \in \mathcal{B}$ such that for all $\overline{Q} \in \mathcal{B}$ we have

$$\overline{P} + \overline{Q} \subseteq \overline{S}.$$

[lh_5] For every $\overline{P} \in \mathcal{B}$ and $\lambda > 0$ there exists $\overline{Q} \in \mathcal{B}$ such that

$$\lambda \overline{P} \subseteq \overline{Q}.$$

Proof. We will show that the family

$$\mathcal{B} = \{\bar{P} \overset{not.}{=} K[P] : P \in \chi(0)\}$$

satisfies the conditions of the theorem. Thus, in order to prove $[lh_1]$, it is sufficient to see that $0 \notin \overline{P}$ because $0 \notin P$ for all $P \in \chi(0)$. Farther, if $\overline{P} = K[P]$ and $\overline{Q} = K[Q]$, we have $L = P \cup Q \in \chi(0)$, hence $\overline{L} = K[L]$ is the perspective which satisfies $[lh_2]$. Condition $[lh_3]$ is assured by Lemma 7, with the completion mentioned in Remark 8.

Because $\mathcal{B} \subseteq \chi(0)$, and the translation is discrete, we have

$$\overline{P} + \overline{Q} = \cup\{y + \overline{Q} \in \chi(y) : y \in \overline{P}\}.$$

By $[h_4]$ there will exist $S \in \chi(0)$ such that $\overline{P} + \overline{Q} \subseteq S$. The fourth condition is verified for $\overline{S} = K[S]$.

The last property is a consequence of the discreteness of dilation D_λ at 0. In fact, for each $\overline{P} \in \mathcal{B} \subseteq \chi(0)$ we can find $Q \in \chi(0)$ such that $\lambda\overline{P} \subseteq Q$. Then we consider $\overline{Q} = K[Q] \in \mathcal{B}$, so that $\lambda\overline{P} \subseteq \overline{Q}$. ■

Theorem 10 *Let W be a real linear space and let $\mathcal{B} \subset \mathcal{P}(W)$ be such that the conditions $[lh_1] - [lh_5]$ from the above Theorem are fulfilled. If we note*

$$\chi(0) = \{P \subset W : P \subseteq \overline{P} \ \ for \ \ some \ \ \overline{P} \in \mathcal{B}\}, \ \ and$$

$$\chi(x) = x + \chi(0) \ \ for \ \ all \ \ x \in W,$$

then χ is a linear horistology on W.

Proof. Obviously, $\chi(0)$ verifies $[h_2]$ and $[h_3]$, being the ideal generated by $\mathcal{B}$. It is also easily seen that $[h_1]$ is assured by $[lh_1]$.

In order to prove $[h_4]$, we may remark that if we take $y \in P \subseteq \overline{P} \in \mathcal{B}$ and $Q \in \chi(y) = y + \chi(0)$, then $\{y\} \in \chi(0)$, and $Q = y + \widetilde{Q}$ for some $\widetilde{Q} \in \chi(0)$. According to $[lh_4]$, there exists $\overline{S} \in \mathcal{B}$ such that

$$Q = y + \widetilde{Q} \subseteq \overline{S}.$$

Consequently, χ is a horistology on W. As usually, we note

$$K = K(\chi) = \{(x, y) \in W^2 : \exists\overline{P} \in \mathcal{B} \ \ such \ \ that \ \ y \in x + \overline{P}\} \cup \delta.$$

Now, let us prove that χ is linear, i.e. the conditions in Definition 2 are fulfilled. In fact, condition (i) is contained in the construction of $\chi(x)$. The discreteness at $(0, 0)$ of the addition A follows from $[lh_4]$. Similarly, the discreteness of the dilation is expressed in $[lh_5]$.

In order to prove (iv), we primarily note that if $x \in \dot{K}[0]$, then for every $\varepsilon > 0$ we have $\{\varepsilon x\} \in \chi(0)$, because $\varepsilon x \in \overline{P}$ holds for some $\overline{P} \in \mathcal{B}$. If

$$Q_\varepsilon = \{\lambda \in \mathbb{R} : \lambda \geq \varepsilon > 0\}$$

is an arbitrary perspective of the origin of $\mathbb{R}$, then

$$E_x(Q_\varepsilon) = \{\frac{\lambda}{\varepsilon}\varepsilon x : \frac{\lambda}{\varepsilon} \geq 1\} \subseteq \overline{P}$$

since $\overline{P}$ is extensive.

Finally, for (v), let us consider $Q_\varepsilon = \{\lambda \in \mathbb{R} : \lambda \geq \varepsilon > 0\} \in \sigma(0)$ and $P \in \chi(0)$. It is easy to see that

$$M(Q_\varepsilon, P) = \{\frac{\lambda}{\varepsilon}\varepsilon x : \lambda \geq \varepsilon,\ \varepsilon x \in \varepsilon P\} = e(\varepsilon P) \subseteq \varepsilon\overline{P}$$

holds for some $\overline{P} \in \mathcal{B}$. According to $[lh_5]$, $\varepsilon\overline{P} \in \chi(0)$, hence the multiplication M is discrete at $(0, 0)$. ■

Remark 11 *The conditions in Definition 2 are formulated with the aim to allow a very detailed analysis, but they are not independent. For example, if* $\lambda > 0$, *then* $\{\lambda\} \in \sigma(0)$, *and for arbitrary* $P \in \chi(0)$ *we have*

$$D_\lambda(P) = M(\{\lambda\}, P),$$

hence (v) *implies* (iii)*. Similarly, if* $x \in \dot{K}[0]$*, then* $\{x\} \in \chi(0)$*, and for all* $Q \in \sigma(0)$ *it follows that*

$$E_x(Q) = M(Q, \{x\}),$$

i.e. (v) *implies* (iv)*.*

On the other hand, (i) *is essential in obtaining* (1)*, i.e. even in the definition of* χ*, but we cannot deduce it from* (ii)*. In fact,* $\{x\} \in \chi(0)$ *holds if and only if* $x \in \dot{K}[0]$*, hence the discreteness of* A *at* $(0, 0)$ *involves only particular elements* $x \in W$*, while in* (i) *we need* x *to be arbitrary.*

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